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BERGMAN METRIC ON A DOMAIN OF THULLEN TYPE

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0. Introduction. In this paper we shall study the Bergman metric on a complete Reinhardt domain $D(p_1,p_2)$ in $\mathbb{C}^2$ given by

$$D(p_1,p_2):= \{(z^1,z^2) \in \mathbb{C}^2;\ |z^1|^{2/p_1} + |z^2|^{2/p_2} < 1\},$$

where p_1, $p_2 > 0$. For $p > 0$ with $p \neq 1$, the domain $D_p := D(1,p)$ was first studied by Thullen [10]. So, we call $D(p_1,p_2)$ a domain of Thullen type (The domain $D(2,2)$ is studied by Kritikos [8]).

In §1, we generally note that the domains $\{z \in \mathbb{C}^n;\ \sum_{j=1}^n |z^j|^{2/p_j} < 1\}$ $(p_j > 0)$ are pseudoconvex. This follows from Hölder's inequality.

Let u_p (resp. ℓ_p) be the supremum (resp. the infimum) of the holomorphic sectional curvature of the Bergman metric on D_p for $p \geqq 0$, where $D_0 := \{z \in \mathbb{C}^2;\ |z^1| < 1,\ |z^2| < 1\}$. In [1], it is shown that

$$\begin{cases} u_p = -2(2+11p+15p^2+8p^3)/(2+p)(1+3p)(4+5p) \\ \ell_p = -(1+4p+p^2)/(1+2p)^2 \end{cases}$$

for $0 \leqq p \leqq 1$. Using the same arguments as in [1], we obtain that

$$\begin{cases} u_p = -2/(2+p) \\ \ell_p = -2(2+11p+15p^2+8p^3)/(2+p)(1+3p)(4+5p) \end{cases}$$

for $p > 1$ (Theorem 2.6).

In §3, we shall study the holomorphic sectional curvature at the origin 0 of the Bergman metric on $D(N,N)$ with $N \in \mathbb{N}$. If $U^N(0)$ and $L^N(0)$ are the maximum and minimum, respectively, of the curvature at 0, then $\lim_{N\to+\infty} U^N(0) = 2$ and $\lim_{N\to+\infty} L^N(0) = -\infty$. This shows that the image of the curvature for the domain $D(N,N)$ includes an arbitrarily given interval $[\alpha,\beta] \subset (-\infty,2)$ for some sufficiently large N (Theorem 3.6).

In §4, we shall study the Ricci curvature of the Bergman metric on D_p. Let $p > 0$ with $p \neq 1$. The Bergman metric on D_p is not Einsteinian, but it is shown that there exists a real hypersurface M in D_p such that the Ricci tensor of the Bergman metric is homothetic to the metric tensor at each point of M (Proposition 4.2).

1. Basic facts. Let $D(p_1,\cdots,p_n) := \{z \in \mathbb{C}^n; \sum_j |z^j|^{2/p_j} < 1\}$ $(p_j > 0)$. Then the assertion

(S) $D(p_1,\cdots,p_n)$ is pseudoconvex

follows from Hölder's inequality. Indeed, since

$D(p_1,\cdots,p_n)$ is a complete Reinhardt domain, (S) is equivalent to the assertion

$$\text{(S')}\quad V(p_1,\cdots,p_n) := \{u \in \mathbb{R}^n;\ \textstyle\sum_j \exp(2u^j/p_j) < 1\}$$

is a convex subset of $\mathbb{R}^n$.

So, we shall show (S'). Let $\lambda,\ \mu \in (0,1)$ with $\lambda + \mu = 1$, and let $u,\ v \in V(p_1,\cdots,p_n)$. Then

$$\begin{aligned}
&\textstyle\sum_j \exp(2(\lambda u^j + \mu v^j)/p_j) \\
&= \textstyle\sum_j \exp(2\lambda u^j/p_j)\exp(2\mu v^j/p_j) \\
&\leqq \left(\textstyle\sum_j \exp(2u^j/p_j)\right)^{1/\lambda}\left(\textstyle\sum_j \exp(2v^j/p_j)\right)^{1/\mu}.
\end{aligned}$$

The last inequality follows from Hölder's inequality, since $\lambda + \mu = 1$. So, we have $\lambda u + \mu v \in V(p_1,\cdots,p_n)$, as desired.

Incidentally, the assertion (S) is proved also as follows: $D(p_1,\cdots,p_n)$ is rewritten as a Hartogs domain

$$\begin{aligned}
D(p_1,\cdots,p_n) = \{z \in \mathbb{C}^n;\ &(z^1,\cdots,z^{n-1}) \in D(p_1,\cdots,p_{n-1}), \\
&|z^n| < \exp(-\varphi_n(z^1,\cdots,z^{n-1}))\},
\end{aligned}$$

where

$$\varphi_n(z^1,\cdots,z^{n-1}) := -\frac{p_n}{2}\log\left(1 - \textstyle\sum_{j=1}^{n-1} |z^j|^{2/p_j}\right).$$

Since the function $(-\infty,1) \ni t \mapsto -\log(1-t) \in \mathbb{R}$ is increasing and convex, and since $\mathbb{C}^{n-1} \ni (z^1,\cdots,z^{n-1}) \mapsto \sum_{j=1}^{n-1} |z^j|^{2/p_j} \in \mathbb{R}$ is plurisubharmonic, φ_n is plurisubharmonic on $D(p_1,\cdots,p_n)$ (cf. Hörmander [5; Theorem 1.6.7]). So, by induction on n, we can show that $D(p_1,\cdots,p_n)$ is pseudoconvex (cf. Lelong [9; Proposition 15]).

Now, we review the Bergman metric on a bounded domain. Let D be a bounded domain in $\mathbb{C}^n$, and $K(z,\bar{\zeta})$ the Bergman kernel of D. The <u>Bergman metric</u> g on D is given by

$$g := 2 \textstyle\sum_{a,b} g_{a\bar{b}}\, dz^a \cdot d\bar{z}^b,$$

where

$$(1.1) \qquad g_{a\bar{b}} := (\partial_a \overline{\partial_b}).\log k, \quad k(z) := K(z,\bar{z})$$

(The function k on D is called the <u>Bergman function</u> of D). The <u>Riemann curvature tensor</u> of g is given by

$$(1.2) \quad R_{a\bar{b}c\bar{d}} := (\partial_c \overline{\partial_d}).g_{a\bar{b}} - \textstyle\sum_{e,\bar{f}} g^{e\bar{f}} (\partial_c . g_{a\bar{f}})(\overline{\partial_d} . g_{e\bar{b}}),$$

where $(g^{e\bar{f}})$ is the inverse matrix of $(g_{a\bar{b}})$ in the sense that $\sum_b g_{a\bar{b}}\, g^{c\bar{b}} = \delta^c_a$. For a_j, $b_j \in \{1,\cdots,n\}$, we denote by $M_{a_1\cdots a_p \bar{b}_1\cdots\bar{b}_q}$ the function

$$\frac{1}{k^2} \begin{vmatrix} k & (\partial_{a_1}\cdots\partial_{a_p}).k \\ (\overline{\partial_{b_1}\cdots\partial_{b_q}}).k & (\partial_{a_1}\cdots\partial_{a_p}\overline{\partial_{b_1}\cdots\partial_{b_q}}).k \end{vmatrix}.$$

Then the formulas (1.1) and (1.2) are rewritten as follows (cf. Kobayashi [7; p.275]):

$$(1.3) \qquad g_{a\bar{b}} = M_{a\bar{b}},$$

$$
(1.4)\qquad
\begin{aligned}
R_{a\bar{b}c\bar{d}} &= -(g_{a\bar{b}}\, g_{c\bar{d}} + g_{a\bar{d}}\, g_{c\bar{b}}) + \hat{R}_{a\bar{b}c\bar{d}},\\
\hat{R}_{a\bar{b}c\bar{d}} &:= M_{ac\bar{b}\bar{d}} - \textstyle\sum_{e,f} g^{e\bar{f}}\, M_{ac\bar{f}}\, M_{e\bar{b}\bar{d}}.
\end{aligned}
$$

For $X = \sum_a v^a(\partial_a)_q \in T_q(D)-\{0\}$ $(q \in D)$, we denote by $HSC(q;X)$ the <u>holomorphic sectional curvature</u> of g in the direction X, i.e.,

$$
(1.5)\qquad
\begin{aligned}
HSC(q;X) &:= -\textstyle\sum R_{a\bar{b}c\bar{d}}(q)\, v^a\bar{v}^b v^c\bar{v}^d \,/\, g(X,\bar{X})^2\\
&= 2 - \textstyle\sum \hat{R}_{a\bar{b}c\bar{d}}(q)\, v^a\bar{v}^b v^c\bar{v}^d \,/\, g(X,\bar{X})^2.
\end{aligned}
$$

<u>DEFINITION 1.1</u>. Given $q \in D$, set

$$U(q) = U_D(q) := \max\{HSC(q;X);\ X \in T_q(D)-\{0\}\},$$
$$L(q) = L_D(q) := \min\{HSC(q;X);\ X \in T_q(D)-\{0\}\}.$$

Furthermore, define the quantities

$$u = u_D := \sup U_D(D), \qquad \ell = \ell_D := \inf L_D(D).$$

The functions U_D and L_D are biholomorphic invariants, and so are the quantities u_D and ℓ_D (cf. [1; Proposition in §3]). It is well-known that $U_D < 2$ for any bounded domain D (cf. Fuks [4; p.525], Kobayashi [7; Theorem 4.4]).

The <u>Ricci tensor</u> of g is given by

$$
(1.6)\qquad
\begin{aligned}
R_{a\bar{b}} &:= -\textstyle\sum_{c,d} g^{c\bar{d}}\, R_{c\bar{d}a\bar{b}}\\
&= (n+1)g_{a\bar{b}} - \textstyle\sum_{c,d} g^{c\bar{d}}\, \hat{R}_{c\bar{d}a\bar{b}}.
\end{aligned}
$$

For $X = \sum_a v^a(\partial_a)_q \in T_q(D)-\{0\}$, we denote by $RC(q;X)$ the Ricci curvature of g in the direction X, i.e.,

$$(1.7) \qquad RC(q;X) := \sum_{a,b} R_{a\bar{b}}\, v^a \bar{v}^b \,/\, g(X,\bar{X}).$$

It is well-known that $RC < n+1$ in general (cf. [7; Theorem 4.4]), and that $RC = -1$ if D is homogeneous, i.e., if the group of all biholomorphic transformations of D acts on D transitively (cf. [7; Theorem 4.1]). When $RC(q;X)$ is a function only of $q \in D$, i.e., independent on $X \in T_q(D)-\{0\}$ for every $q \in D$, the metric g is called Einsteinian. So, we may use the following.

DEFINITION 1.2. A point $q \in D$ is called Einsteinian if $RC(q;\cdot)$ is constant on $T_q(D)-\{0\}$.

Now, suppose D is a complete Reinhardt domain, and consider $\Omega = \Omega_D := \{(|z^1|,\cdots,|z^n|) \in \mathbb{R}^n_+;\ z \in D\}$ and

$$(1.8) \qquad a_I := \left(\int_\Omega (r^1)^{2i_1+1} \cdots (r^n)^{2i_n+1}\, dr^1 \ldots dr^n\right)^{-1}$$

for $I = (i_1,\cdots,i_n) \in \mathbb{Z}^n_+$. Then the Bergman function k of D is given by

$$(1.9) \qquad k(z) = \frac{1}{(2\pi)^n} \sum_I a_I\, z^I \bar{z}^I \qquad (z \in D).$$

For the domain $D(p_1,\cdots,p_n)$ introduced in the first part of this section, it is calculated that

$$(1.10) \qquad a_I^{-1} = \frac{p_1 \cdots p_n}{2^n}\, \frac{\Gamma(p_1(i_1+1)) \cdots \Gamma(p_n(i_n+1))}{\Gamma(\sum_j p_j(i_j+1) + 1)}$$

(cf. Ise [6; p.520], D'Angelo [3; Lemma 2]).

2. Holomorphic sectional curvature of the Bergman metric on D_p. The domain $D_p = D(1,p)$ introduced in §0 is rewritten as follows:

$$(2.1)\quad D_p = \{z \in \mathbb{C}^2;\ |z^1| < 1,\quad |z^2|^2 < (1-|z^1|^2)^p\},$$

which makes sense as an unbounded, non-complete Reinhardt domain also when $p < 0$. First we note the following.

LEMMA 2.1. *For any real* p, *the group of all biholomorphic transformations of* D_p *includes the group* G *of mappings*

$$\begin{cases} w^1 = \lambda(z^1+\alpha)/(1+\bar{\alpha}z^1) \\ w^2 = \mu(1-|\alpha|^2)^{p/2}(1+\bar{\alpha}z^1)^{-p}z^2, \end{cases}$$

where $\lambda, \mu, \alpha \in \mathbb{C}$ *with* $|\lambda| = |\mu| = 1$, $|\alpha| < 1$. *Furthermore, the orbit space* $G(0,\omega)$ *of a point* $(0,\omega) \in D_p$ *with* $0 \leqq \omega < 1$ *is given by*

$$\{z \in D_p;\ \omega^2 = |z^2|^2(1-|z^1|^2)^{-p}\},$$

and it holds that $\bigsqcup_{0\leqq\omega<1} G(0,\omega) = D_p$.

By virtue of Lemma 2.1, for the purpose of studying functions on D_p which are biholomorphic invariant, such as U and L, it is enough to examine the values only at points $(0,\omega)$ with $0 \leqq \omega < 1$.

From now on, we are concerned with domains D_p

only of $p \geqq 0$ in the expression (2.1). By the use of (1.9) and (1.10), the Bergman function k of D_p is calculated. As a result, we have

$$(2.2) \qquad k(z) = \frac{1}{\pi^2} \frac{(1+p)(1-|z^1|^2)^p - (1-p)|z^2|^2}{((1-|z^1|^2)^p - |z^2|^2)^3 (1-|z^1|^2)^{2-p}}$$

(cf. Bergman [2; p.21]). As in [1; Lemma 3], we make use of the variables

$$(2.3) \qquad \begin{cases} r = r(p) := (1-p)/(1+p) & (p \geqq 0) \\ t = t(\omega) := (1-\omega^2)/(1-r\omega^2) & (0 \leqq \omega < 1), \end{cases}$$

and the functions

$$(2.4) \qquad \begin{cases} \alpha := 3 + rt^2 \\ \beta := 3 - rt^2 \\ A := 6 + 4rt^2 + (1+r)rt^3 \\ B := 2(9 + 3rt^2 - 3(1+r)rt^3 + 2r^2t^4)/\alpha \\ C := 3(6 - 6rt^2 + (1+r)rt^3)/\beta . \end{cases}$$

The graph of the function $r = r(p)$ is as in Figure 1. When $p > 0$, i.e., when $-1 < r < 1$, it follows that

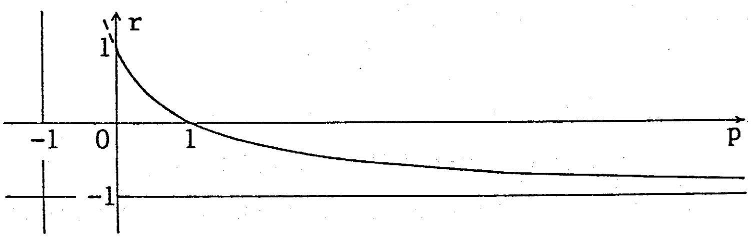

Figure 1. The graph of $r = r(p)$

$$(2.5)_1 \qquad k(0,\omega) = 2(1-rt)^2/\pi^2(1+r)(1-r)^2t^3,$$

$$(2.5)_2 \qquad \begin{cases} g_{1\bar{1}}(0,\omega) = \alpha/(1+r)t \\ g_{1\bar{2}}(0,\omega) = 0 \\ g_{2\bar{2}}(0,\omega) = \beta(1-rt)^2/(1-r)^2t^2, \end{cases}$$

$$(2.5)_3 \qquad \begin{cases} R_{1\bar{1}1\bar{1}}(0,\omega) = 4A/(1+r)^2t^2 \\ R_{1\bar{1}2\bar{2}}(0,\omega) = 2(1-rt)^2B/(1+r)(1-r)^2t^3 \\ R_{2\bar{2}2\bar{2}}(0,\omega) = 4(1-rt)^4C/(1-r)^4t^4 \\ R_{1\bar{1}1\bar{2}}(0,\omega) = 0 \\ R_{1\bar{2}1\bar{2}}(0,\omega) = 0 \\ R_{1\bar{2}2\bar{2}}(0,\omega) = 0 \end{cases}$$

(cf. [1; p.5]). We obtain the following.

<u>LEMMA 2.2</u>. <u>For</u> D_p <u>with</u> $p > 0$ (<u>cf</u>. (2.1)), <u>the values of</u> $U_p := U_{D_p}$ <u>and</u> $L_p := L_{D_p}$ <u>at</u> $(0,\omega)$ <u>with</u> $0 \leqq \omega < 1$ <u>are given by</u>

$$U_p(0,\omega) = 2 - 4 \min \{Ax^2 + 2Bxy + Cy^2;\ x,\ y \geqq 0,\ \alpha x + \beta y = 1\},$$

$$L_p(0,\omega) = 2 - 4 \max \{Ax^2 + 2Bxy + Cy^2;\ x,\ y \geqq 0,\ \alpha x + \beta y = 1\}$$

<u>where</u> α, β, A, B, <u>and</u> C <u>are the values defined in</u> (2.4).

<u>PROOF</u>. When $0 < p \leqq 1$, the assertions are the same as [1; Lemma 3]. But, by the formulas $(2.5)_2$ and $(2.5)_3$, the statements hold also for $p > 1$. Q.E.D.

We now suppose $p > 1$. It follows that

(2.6) $$-1 < r < 0.$$

From (2.4), we obtain

(2.7) $$\begin{cases} C\alpha - B\beta = rt^3E_1/\alpha\beta \\ A\beta - B\alpha = rt^3E_2, \end{cases}$$

where

(2.8) $$\begin{cases} E_1 := 9^2(1+r) - 12\cdot 9rt - 2\cdot 9(1+r)rt^2 + 9(1+r)r^2t^4 - 4r^3t^5 \\ E_2 := 9(1+r) - 8rt - (1+r)rt^2. \end{cases}$$

So, it follows from (2.6) that

(2.9) $$\alpha > 0, \quad \beta > 0, \quad C\alpha - B\beta < 0, \quad A\beta - B\alpha < 0.$$

In general, we can show the following.

LEMMA 2.3. *Let* α, β, A, B, *and* C *be real numbers with the properties* (2.9), *and let* $f(x,y) := Ax^2 + 2Bxy + Cy^2$, $g(x,y) := \alpha x + \beta y$. *Then it holds that*

$$\max\{f(x,y);\ x,\ y \geqq 0,\ \ g(x,y) = 1\} = \frac{AC - B^2}{A\beta^2 - 2B\alpha\beta + C\alpha^2},$$

$$\min\{f(x,y);\ x,\ y \geqq 0,\ \ g(x,y) = 1\} = \min\{A/\alpha^2,\ C/\beta^2\}$$

(*Note that* $A\beta^2 - 2B\alpha\beta + C\alpha^2 = \beta(A\beta - B\alpha) + \alpha(C\alpha - B\beta) < 0$). *Furthermore, when* $C = A$ *and* $\beta = \alpha$, *it holds that*

$$\max\{f(x,y);\ x,\ y \geqq 0,\ \ g(x,y) = 1\} = (A+B)/2\alpha^2,$$

$$\min\{f(x,y);\ x,\ y \geqq 0,\ \ g(x,y) = 1\} = A/\alpha^2.$$

The last two formulas make sense also when $A = B$.

From Lemmas 2.2 and 2.3, and from (2.9), we obtain the following.

THEOREM 2.4. *For the domain* D_p *with* $p > 1$, *the values of* $U_p := U_{D_p}$ *and* $L_p := L_{D_p}$ *at* $(0,\omega)$ *with* $0 \leqq \omega < 1$ *are given by*

$$U_p(0,\omega) = 2 - 4 \min \{A/\alpha^2, \quad C/\beta^2\},$$

$$L_p(0,\omega) = 2 - 4F/\alpha^2 E,$$

where α, β, A, *and* C *are as in* (2.4), *and*

$$\begin{aligned} E &:= E_1 + \beta^2 E_2 \qquad (\textit{cf.}\ (2.8)) \\ &= 2\cdot 9^2(1+r) - 20\cdot 9rt - 9^2(1+r)rt^2 + 48r^2t^3 + 24(1+r)r^2t^4 \\ &\quad - 12r^3t^5 - (1+r)r^3t^6, \end{aligned}$$

$$\begin{aligned} F &:= 12\cdot 9^2(1+r) - 120\cdot 9rt + 2\cdot 9^2(1+r)rt^2 - 3\cdot 9(3(1+r)^2+16r)rt^3 \\ &\quad + 8\cdot 9(1+r)r^2t^4 + 2\cdot 9(3(1+r)^2-4r)r^2t^5 - 6\cdot 9(1+r)r^3t^6 \\ &\quad + (3(1+r)^2+16r)r^3t^7. \end{aligned}$$

We shall show the following.

THEOREM 2.5. *For* D_p *with* $p > 1$, *the following hold*: (i) $\lim_{\omega\to 1} U_p(0,\omega) = \lim_{\omega\to 1} L_p(0,\omega) = -2/3$.
(ii) $U_p(0,\omega)$ *is strictly decreasing with respect to* ω.
(iii) $L_p(0,\omega)$ *is strictly increasing with respect to* ω.

PROOF. (i): Since $t(\omega) \to 0$ as $\omega \to 1$, the assertion follows from Theorem 2.4.

(ii): We have

$$\frac{\partial}{\partial t} C/\beta^2 = 3rt^2(3(1+r) - 8rt + (1+r)rt^2)/\beta^4,$$
$$\frac{\partial}{\partial t} A/\alpha^2 = rt^2(9(1+r) - 8rt - (1+r)rt^2)/\alpha^3.$$

So, it follows from (2.6) that $\frac{\partial}{\partial t} A/\alpha^2$, $\frac{\partial}{\partial t} C/\beta^2 < 0$; therefore, the desired assertion follows from Theorem 2.4, since $t(\omega)$ is strictly decreasing with respect to ω.

(iii) Let E and F be as in Theorem 2.4. We have

$$\frac{\partial}{\partial t} F/\alpha^2 E = rt^2 M/\alpha^3 E^2,$$

where

$$M := 3\cdot 9^2 N_1 + 2\cdot 9^2 r^3 t^5 N_2 + 9r^4 t^8 N_3 + 3r^6 t^{11} N_4,$$

$$N_1 := -6\cdot 9^2(1+r)^3 + 168\cdot 9(1+r)^2 rt + 15(45(1+r)^2 - 112r)(1+r)rt^2$$
$$- 16(123(1+r)^2 - 40r)r^2t^3 - (141(1+r)^2 - 1808r)(1+r)r^2t^4,$$

$$N_2 := 24(13(1+r)^2 - 32r) + 3(31(1+r)^2 + 16r)(1+r)t$$
$$- 16(9(1+r)^2 + 8r)rt^2,$$

$$N_3 := -16\cdot 9(3(1+r)^2 - 10r) + 8(75(1+r)^2 - 64r)rt$$
$$+ (21(1+r)^2 - 304r)(1+r)rt^2,$$

$$N_4 := -16(3(1+r)^2 - 8r) + (3(1+r)^2 + 16r)(1+r)t.$$

It is trivially seen that $N_1, N_3 < 0$ (cf. (2.6)). We shall show that $N_2 > 0$ and $N_4 < 0$. For this, we set $N_2 =: a(r) + b(r)t + c(r)t^2$, $N_4 =: d(r) + e(r)t$, and consider two functions $f_r(u) := a(r)u^2 + b(r)u + c(r)$ and $g_r(u) := d(r)u + e(r)$. It is easily seen that $f_r(1) = a(r) + b(r) + c(r)$, $f_r^{(1)}(1) = 2a(r) + b(r)$, $f_r^{(2)} = 2a(r) > 0$, and $g_r(1) = d(r) + e(r)$, $g_r^{(1)} = d(r) < 0$ for any r with (2.6). So, it follows that $f_r > 0$ and $g_r < 0$ on the interval $[1,+\infty)$; therefore we have $N_2 > 0$ and $N_4 < 0$. Combining these facts with the facts $N_1, N_3 < 0$, we obtain $M > 0$. Thus the desired assertion follows from Theorem 2.4. Q.E.D.

We also obtain the following.

THEOREM 2.6. *For D_p with $p > 1$, the quantities $u_p := u_{D_p}$ and $\ell_p := \ell_{D_p}$ are given by*

$$u_p = U_p(0,0) = 2 - 4(2+r)/(3+r) = -2/(2+p),$$

$$\ell_p = L_p(0,0) = 2 - 4(36-13r-3r^2)/(2-r)(9-r)(3+r)$$

$$= -2(2+11p+15p^2+8p^3)/(2+p)(1+3p)(4+5p).$$

PROOF. Since $A/\alpha^2|_{t=1} = (2+r)/(3+r)$, $C/\beta^2|_{t=1} = 3(2-r)/(3-r)^2$, $E|_{t=1} = (2-r)(9-r)(3-r)(3+r)$, $F|_{t=1} = (3-r)(3+r)^2(36-13r-3r^2)$, the desired assertion follows from Theorems 2.4 and 2.5. Q.E.D.

REMARK 2.7. For D_p with $0 \leqq p \leqq 1$, it is shown in [1; Theorem 3] that

$$u_p = 2 - 4(36-13r-3r^2)/(2-r)(9-r)(3+r)$$

$$= -2(2+11p+15p^2+8p^3)/(2+p)(1+3p)(4+5p)$$

$$\ell_p = 2 - 12(2-r)/(3-r)^2 = -(1+4p+p^2)/(1+2p)^2.$$

The graphs of the functions $y = u(r)$ and $y = \ell(r)$ $(-1 < r \leqq 1)$ are as in Figure 2, where $u(r(p)) = u_p$, $\ell(r(p)) = \ell_p$ $(p \geqq 0)$ (also cf. Figure 1).

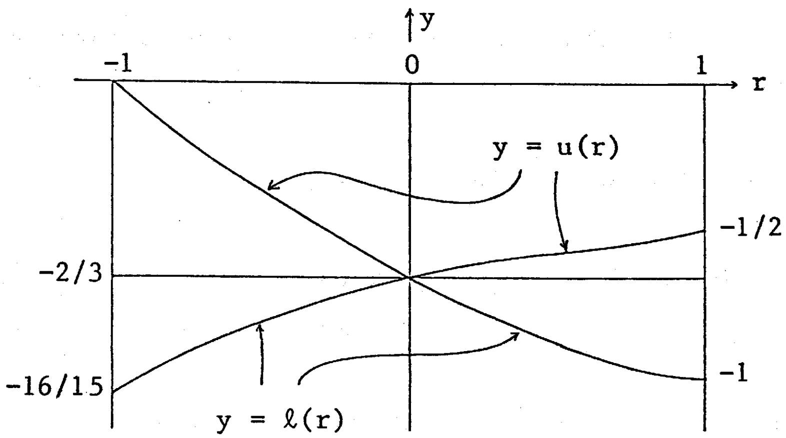

Figure 2. The graphs of $y = u(r)$, $y = \ell(r)$

We shall show the following.

PROPOSITION 2.8. *Let* $0 \leqq p < p'$. *Then* $u_p \neq u_{p'}$ *or* $\ell_p \neq \ell_{p'}$. *Especially* D_p *is not biholomorphic to* $D_{p'}$.

PROOF. From Theorem 2.6 and Remark 2.7, it is easily seen that $\ell_p < \ell_{p'}$ when $p' \leqq 1$, and that $u_p < u_{p'}$ when $p \geqq 1$. So, we suppose that $p < 1 < p'$, and that $u_p = u_{p'}$, $\ell_p = \ell_{p'}$. Set $r := r(p)$, $r' := r(p')$ (cf. (2.3)). Then

(2.10) $$-1 < r' < 0 < r \leqq 1.$$

From Theorem 2.6 and Remark 2.7, we have

$$\begin{cases} u_{p'} = 2 - 4(2+r')/(3+r') \\ \ell_{p'} = 2 - 4(36-13r'-3r'^2)/(2-r')(9-r')(3+r'), \end{cases}$$

$$\begin{cases} u_p = 2 - 4(36-13r-3r^2)/(2-r)(9-r)(3+r) \\ \ell_p = 2 - 4\cdot 3(2-r)/(3-r)^2. \end{cases}$$

It follows from our assumption that

(2.11) $$\frac{36-13r-3r^2}{(2-r)(9-r)(3+r)} = \frac{2+r'}{3+r'},$$

(2.12) $$\frac{3(2-r)}{(3-r)^2} = \frac{36-13r'-3r'^2}{(2-r')(9-r')(3+r')}.$$

From (2.11), we have

(2.13) $$r' = - rG/F,$$

where

(2.14) $$F := 18-2r-5r^2+r^3, \quad G := 9-7r+2r^2.$$

Substituting (2.13) to (2.12), we obtain

(2.15) $$\begin{aligned} 18(3-2r)F^3 &= (27-33r+13r^2)F^2G \\ &\quad + 3(7-2r-r^2)rFG^2 + 3(2-r)r^2G^3. \end{aligned}$$

Let $x := 1/r - 1$. Then $x \geqq 0$ by (2.10). Furthermore, (2.14) and (2.15) are rewritten as follows:

$$F = \tilde{F}(1+x)^{-3}, \quad G = \tilde{G}(1+x)^{-2},$$

$$\begin{aligned} 18(1+3x)\tilde{F}^3 &= (7+21x+27x^2)\,\tilde{F}^2\tilde{G} \\ &+ 3(4+12x+7x^2)\,\tilde{F}\tilde{G}^2 \\ &+ 3(1+2x)(1+x)\,\tilde{G}^3, \end{aligned} \tag{2.16}$$

where $\tilde{F}:= 12+45x+52x^2+18x^3$, $\tilde{G}:= 4+11x+9x^2$. But since the coefficients of four polynomials $6(1+3x)\tilde{F} - (7+21x+27x^2)\tilde{G}$, $2(1+3x)\tilde{F} - (4+12x+7x^2)\tilde{G}$, $2(1+3x)\tilde{F} - (1+2x)(1+x)\tilde{G}$, $\tilde{F}-\tilde{G}$ are all positive, the equation (2.16) has no zero in the interval $[0,+\infty)$. This is a contradiction. Q.E.D.

REMARK 2.9. Let k be the Bergman function of a bounded domain D, and let $g = 2\sum g_{a\bar{b}}\, dz^a \cdot d\bar{z}^b$ be the Bergman metric on D. Then the function $\det(g_{a\bar{b}})/k$ is a biholomorphic invariant. For the domain D_p with $p \geqq 0$, it follows from $(2.5)_1$, $(2.5)_2$ (and from direct calculations for $p = 0$) that

$$(\det(g_{a\bar{b}})/k)(0,\omega) = \begin{cases} 4\pi^2, & p = 0 \\ \pi^2(9-r(p)^2t^4)/2, & p > 0; \end{cases}$$

therefore we have

$$m_1(p) := \sup(\det(g_{a\bar{b}})/k) = \begin{cases} 4\pi^2, & p = 0 \\ 9\pi^2/2, & p > 0, \end{cases}$$

$$m_2(p) := \inf(\det(g_{a\bar{b}})/k) = \pi^2(9-r(p)^2)/2.$$

So, the pair (m_1, m_2) of numerical invariants does not distinguish pairs D_{p_1}, D_{p_2} with $r(p_1) = -r(p_2)$, or with $p_1 p_2 = 1$.

3. *Holomorphic sectional curvature of the Bergman metric on* D(N,N). We need the following.

LEMMA 3.1. *Let* D *be a bounded, complete Reinhardt domain in* $\mathbb{C}^2$, *and let* a_I *be the numbers defined in* (1.8). *Then the holomorphic sectional curvature* $HSC(0;X)$ *of the Bergman metric on* D *in the direction* $X \in T_0(D)-\{0\}$ *is given by the formula*

$$HSC(0;X) = 2 - 4\frac{a_{00}(a_{20}x^2 + a_{11}xy + a_{02}y^2)}{(a_{10}x + a_{01}y)^2},$$

where $x := |v^1|^2$, $y := |v^2|^2$, $X = v^1(\partial_1)_0 + v^2(\partial_2)_0$.

PROOF. It follows from (1.3) and (1.4) that

$$g_{a\bar{b}}(0) = \begin{cases} a_{10}/a_{00}, & (a,b) = (1,1) \\ a_{01}/a_{00}, & (a,b) = (2,2) \\ 0, & \text{elsewhere}, \end{cases}$$

$$\hat{R}_{a\bar{b}c\bar{d}}(0) = \begin{cases} 4a_{20}/a_{00}, & (a,b,c,d) = (1,1,1,1) \\ 4a_{02}/a_{00}, & (a,b,c,d) = (2,2,2,2) \\ a_{11}/a_{00}, & \{a,c\} = \{b,d\} = \{1,2\} \\ 0, & \text{elsewhere}. \end{cases}$$

So, the desired formula follows from (1.5). Q.E.D.

Now, for the domain $D(N,N)$ $(N \in \mathbb{N})$ (cf. §0), it follows from (1.10) that

$$
(3.1)\quad \begin{cases} a_{00} = \dfrac{8(2N-1)!}{N(N-1)!(N-1)!} \\ a_{10} = a_{01} = \dfrac{12(3N-1)!}{N(N-1)!(2N-1)!} \\ a_{20} = a_{02} = \dfrac{16(4N-1)!}{N(N-1)!(3N-1)!} \\ a_{11} = \dfrac{16(4N-1)!}{N(2N-1)!(2N-1)!}. \end{cases}
$$

We first note the following.

LEMMA 3.2. *For the values in* (3.1), *we have* $2a_{20} \leqq a_{11}$.

PROOF. We have

$$
a_{11} - 2a_{20} = \frac{16(4N-1)!}{N(2N-1)!(3N-1)!} \times
$$

$$
\{(3N-1)\cdots(3N-N) - 2(2N-1)\cdots(2N-N)\}.
$$

But the factor in the braces is 0 when $N = 1$, and is $2N\{(3N-1)\cdots(2N+1) - (2N-1)\cdots(N+1)\} > 0$ when $N \geqq 2$, as desired. Q.E.D.

By virtue of Lemma 2.3, using Lemmas 3.1 and 3.2, we obtain the following.

Lemma 3.3. *For the domain* $D(N,N)$ *with* $N \in \mathbb{N}$, *the functions* $U^N := U_{D(N,N)}$, $L^N := L_{D(N,N)}$ *are given at* 0 *by the formulas*

$$
U^N(0) = 2 - 4a_{00}a_{20}/a_{10}^{\;2},
$$

$$
L^N(0) = 2 - a_{00}(2a_{20} + a_{11})/a_{10}^{\;2},
$$

where a_{pq} *are the values in* (3.1).

Furthermore, we have the following.

LEMMA 3.4. *For the values in* (3.1), *it holds that*

$$a_{00}a_{20}/a_{10}^{2} = \frac{8}{9}\prod_{j=1}^{N}\left(1 - \left(\frac{N}{3N-j}\right)^{2}\right) < \left(\frac{8}{9}\right)^{N+1},$$

$$a_{00}a_{11}/a_{10}^{2} = \frac{8}{9}\prod_{j=1}^{N}\left(1 + \frac{N}{3N-j}\right) > \frac{8}{9}\left(\frac{4}{3}\right)^{N}.$$

PROOF. From (3.1) we get

$$a_{00}a_{20}/a_{10}^{2} = \frac{8}{9}\,\frac{((2N-1)!)^{3}(4N-1)!}{((3N-1)!)^{3}(N-1)!}$$

$$= \frac{8}{9}\,\frac{(4N-1)\cdots(4N-N)}{(3N-1)\cdots(3N-N)}\,\frac{(2N-1)\cdots(2N-N)}{(3N-1)\cdots(3N-N)}$$

and

$$a_{00}a_{11}/a_{10}^{2} = \frac{8}{9}\,\frac{(2N-1)!\,(4N-1)!}{(3N-1)!\,(3N-1)!}$$

$$= \frac{8}{9}\,\frac{(4N-1)\cdots(4N-N)}{(3N-1)\cdots(3N-N)}.$$

Since $(4N-j)/(3N-j) = 1 + N/(3N-j)$, $(2N-j)/(3N-j) = 1 - N/(3N-j)$, the desired estimates follow. Q.E.D.

From Lemmas 3.3 and 3.4 we get the following.

PROPOSITION 3.5. *Let* U^N *and* L^N *be the functions on* $D(N,N)$ *given in Lemma 3.3. Then* $U^N(0) > 2 - 4(8/9)^{N+1}$ *and* $L^N(0) < 2 - (8/9)(4/3)^N$.

THEOREM 3.6. *For every interval* $[\alpha,\beta] \subset (-\infty,2)$, *there exists a bounded pseudoconvex Reinhardt domain* D *in* $\mathbb{C}^2$ *for which the numerical invariants* ℓ_D, u_D *given in Definition* 1.1 *satisfy* $\ell_D < \alpha$ *and* $u_D > \beta$.

4. Ricci curvature of the Bergman metric on D_p.

According to (1.6), $(2.5)_2$, $(2.5)_3$, the Ricci tensor of the Bergman metric on D_p with $p > 0$ is calculated as follows:

$$\begin{cases} R_{1\bar{1}}(0,\omega) = 3g_{1\bar{1}}(0,\omega) + 2(2A\beta + B\alpha)/\alpha\beta(1+r)t \\ R_{2\bar{2}}(0,\omega) = 3g_{2\bar{2}}(0,\omega) + 2(2C\alpha + B\beta)(1-rt)^2/\alpha\beta(1-r)^2t^2 \\ R_{1\bar{2}}(0,\omega) = 0, \end{cases}$$

where $r = (1-p)/(1+p)$, $t = (1-\omega^2)/(1-r\omega^2)$ $(0 \leqq \omega < 1)$ as in (2.3), and α, β, A, B, and C are the functions given in (2.4).

Now, for $X = v^1(\partial_1)_{(0,\omega)} + v^2(\partial_2)_{(0,\omega)} \in T_{(0,\omega)}(D_p)-\{0\}$, we set $x := |v^1|^2/(1+r)t$, $y := |v^2|^2(1-rt)^2/(1-r)^2t^2$. Then we have

$$\sum g_{a\bar{b}}\, v^a\bar{v}^b = \alpha x + \beta y,$$

$$\sum R_{a\bar{b}}\, v^a\bar{v}^b = 3(\alpha x + \beta y) - 2P\alpha x - 2Q\beta y,$$

where

$$(4.1) \qquad P := (2A\beta + B\alpha)/\alpha^2\beta, \qquad Q := (2C\alpha + B\beta)/\alpha\beta^2.$$

So, it follows from (1.7) that

$$RC((0,\omega);X) = 3 - 2(P\alpha x + Q\beta y)/(\alpha x + \beta y);$$

$$(4.2)\quad \begin{cases} \max RC((0,\omega);\cdot) = 3 - 2\min\{P, Q\} \\ \qquad\qquad\qquad\qquad = -1 + 2\max\{2-P, 2-Q\} \\ \min RC((0,\omega);\cdot) = 3 - 2\max\{P, Q\} \\ \qquad\qquad\qquad\qquad = -1 + 2\min\{2-P, 2-Q\}. \end{cases}$$

Using (2.4) and (4.1), we can calculate as follows:

$$(4.3)_1 \qquad 2 - P = -2r^2t^4(1-t)(1-rt)/\alpha^2\beta,$$

$$(4.3)_2 \qquad 2 - Q = 2r^2t^4R/\alpha^2\beta^3,$$

where

$$R := 3\cdot 9 - 4\cdot 9(1+r)t + 5\cdot 9rt^2 + r^2t^4 - r^3t^6.$$

Furthermore, we have

$$(4.4) \qquad P - Q = (2-Q) - (2-P) = 2r^2t^4S/\alpha^2\beta^3,$$

where

$$(4.5) \qquad \begin{aligned} S := {} & 4\cdot 9 - 5\cdot 9(1+r)t + 48rt^2 + 6(1+r)rt^3 - 4r^2t^4 \\ & - (1+r)r^2t^5. \end{aligned}$$

LEMMA 4.1. *Let* $S_r(t) := S(r,t)$. *Given* $r \in (-1,1)$, *the function* S_r *has only one simple zero* $\varepsilon(r)$ *in the interval* $(0,1)$.

PROOF. We obtain the following:

$$\begin{cases} S_r(1) = -(1-r)(9-r^2) \\ S_r^{(1)}(1) = -45 + 69r - 3r^2 - 5r^3 \end{cases}$$

$$\begin{cases} S_r^{(2)}(1) = 4(33-8r-5r^2)r \\ S_r^{(3)}(1) = 12(3-10r-5r^2)r \\ S_r^{(4)}(1) = -24(9+5r)r^2 \\ S_r^{(5)}(1) = -120(1+r)r^2. \end{cases}$$

$S_r^{(1)}$ (resp. $S_r^{(3)}(1)$) has only one simple zero r_1 (resp. r_2) in the interval $(0,1)$ $(1 < r_2 < 1/2 < r_1 < 1)$. We obtain the following tables of signs of $S_r^{(k)}(0)$ and $S_r^{(k)}(1)$:

Table 1

r	-1 0 1
$S_r(0)$	: + : + :
$S_r^{(1)}(0)$	0 − : − :
$S_r^{(2)}(0)$	: − 0 + :
$S_r^{(3)}(0)$	0 − 0 + :
$S_r^{(4)}(0)$	: − 0 − :
$S_r^{(5)}(0)$	0 − 0 − :

Table 2

r	-1 0 r_2 r_1 1
$S_r(1)$	: − : − : − : − 0
$S_r^{(1)}(1)$	: − : − : − 0 + :
$S_r^{(2)}(1)$	: − 0 + : + : + :
$S_r^{(3)}(1)$	: − 0 + 0 − : − :
$S_r^{(4)}(1)$	: − 0 − : − : − :
$S_r^{(5)}(1)$	0 − 0 − : − : − :

Let $N_r(a)$ be the number of exchanges of sign in the sequence $(S_r(a), S_r^{(1)}(a), \cdots, S_r^{(5)}(a))$. From Tables 1 and 2, we can see that

$$N_r(0) = \begin{cases} 1, & -1 < r \leqq 0 \\ 3, & 0 < r < 1, \end{cases} \qquad N_r(1) = \begin{cases} 0, & -1 < r \leqq 0 \\ 2, & 0 < r < 1; \end{cases}$$

therefore, $N_r(0) - N_r(1) = 1$ for all $r \in (-1,1)$. So, the desired assertion follows from the Fourier's theorem concerning the number of zeros of a real polynomial (cf. [1; Appendices]). Q.E.D.

From Lemma 4.1 and (4.2), we obtain the following.

PROPOSITION 4.2. *Let* $p > 0$ *with* $p \neq 1$. *Then in the subset* $\{(0,\omega);\ 0 \leqq \omega \leqq 1\}$ *of* D_p, *there exists only one Einsteinian point* $(0, \omega(p))$ *of* D_p *(cf. Definition* 1.2). *The value* $\omega(p)$ *is obtained from the equation*

$$\varepsilon(r(p)) = (1 - \omega(p)^2)/(1 - r\omega(p)^2),$$

where $\varepsilon(r)$ *is the unique zero in the interval* $(0,1)$ *of the polynomial* $S_r(t) = S(r,t)$ *given in* (4.5) (*So, the set of all Einsteinian points in* D_p *coincides with the orbit space* $G(0, \omega(p))$, *because of* Lemma 2.1).

Now, differentiating $(4.3)_1$, we obtain a remarkable relation as follows.

Lemma 4.3. *Let* P (*resp.* S) *be as in* (4.1) (*resp.* (4.5)). *Then it holds that*

$$\frac{\partial}{\partial t}(2 - P) = - 2r^2t^3 S/\alpha^3\beta^2.$$

By virtue of Lemmas 4.1 and 4.3, the graphs of the functions $2 - P_r$ and $2 - Q_r$ can be drawn as in Figure 3, where $P_r(t) := P(r,t)$, $Q_r(t) := Q(r,t)$. From Figure 3 and (4.2), we obtain the following.

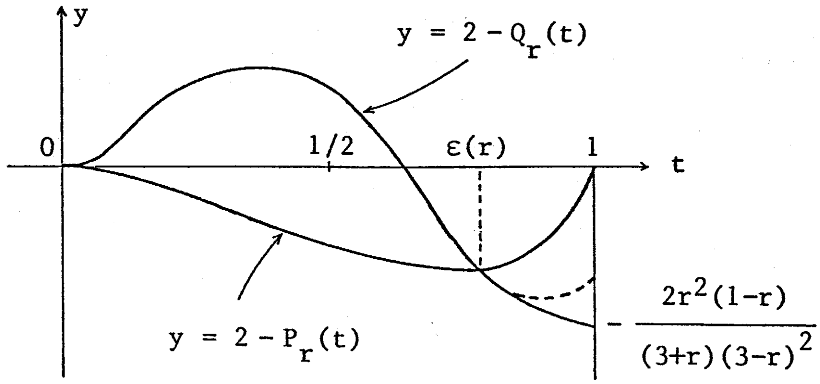

Figure 3. The graphs of $2-P_r$, $2-Q_r$

PROPOSITION 4.4. *Let* $p > 0$ *with* $p \neq 1$, *and let* RC *be the Ricci curvature of the Bergman metric on* D_p. *Then it holds that*

$$\max RC = -1 + 2\max(2-Q_r),$$

$$\min RC = -1 + 2\min(2-Q_r),$$

where $r = (1-p)/(1+p)$.

REFERENCES

[1] K. Azukawa and M. Suzuki; The Bergman metric on a Thullen domain, Nagoya Math. J., 89(1983), 1-11.

[2] S. Bergman; Sur les Fonctions Orthogonales de Plusieurs Variables Complexes avec les Applications à la Théorie des Fonctions Analytiques (Mémorial Sci. Math., 106), Gauthier-Villars, Paris, 1947.

[3] J. P. D'Angelo; A note on the Bergman kernel, Duke Math. J., 45(1978), 259-265.

[4] B. A. Fuks; Ricci curvature of a Bergman metric invariant under biholomorphic mappings, Soviet Math. Dokl., 7(1966), 525-529.

[5] L. Hörmander; An Introduction to Complex Analysis in Several Variables, van Nostrand, Tront-New York-London, 1966.

[6] M. Ise; On Thullen domains and Hirzebruch manifolds, I, J. Math. Soc. Japan, 26(1974), 508-522.

[7] S. Kobayashi; Geometry of bounded domains, Trans. Amer. Math. Soc., 92(1959), 267-290.

[8] N. Kritikos, Über analytische Abbildungen einer Klasse von vierdimensionalen Gebieten, Math. Ann., 99 (1928), 321-341.

[9] P. Lelong, Domains convexes par rapport aux fonctions plurisousharmoniques, J. Analyse Math., 2(1952), 178-208.

[10] P. Thullen, Zu den Abbildungen durch analytische Funktionen mehrerer komplexer Veränderlichen: Die Invarianz des Mittelpunktes von Kreiskörpern, Math. Ann., 104(1931), 244-259.

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